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Exponential decay for the solution of the nonlinear equation induced by the mathematical model in mass production process with one sided spring boundary condition byfeedback control

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ABSTRACT

In this paper we consider the initial-boundary value problem for a nonlinear equation induced with respect to the mathematical models in mass production process with the one sided spring boundary condition by boundary feedback control. We establish the asymptotic behavior of solutions to this problem in time, and give an example and simulation to illustrate our results. Results of this paper are able to apply industrial parts such as a typical model widely used to represent threads, wires, magnetic tapes, belts, band saws, and so on.

Keyword:*Mass production process asymptotic Behavior one sided spring boundary condition boundary feedback control*

I. Introduction

In this paper, we consider the following initial-boundary value problem for a nonlinear Kirchhoff type equation with one sided spring boundary conditions by boundary feedback control with respect to the mathematical models in mass production process :

$$u_{tt}(x,t) - a(x)B(\| \nabla u(t) \|^2)\Delta u(x,t) + Ku(x,t) \quad (1)$$

$$+\lambda u_t(x,t) + \eta u_{xt}(x,t) = 0, (x,t) \in (0,1) \times (0,T);$$

$$u(0,t) = 0, \quad a(1)u_x(1,t) + h_1 u(1,t) = s(t), \; t \in (0,T); \quad (2)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \; x \in (0,1), \quad (3)$$

where K, η and h_1 are given nonnegative constants; λ, and T, the given positive constants; $u_0, u_1, a(x)$, and B, the given functions; and $u(x,t)$, the transversal displacement of the strip at spatial coordinate x and time t. The hypotheses on these functions for our purpose will be specified later. (1) describes the nonlinear vibrations of an elastic string. And also, (2) means both ends attached with springs depends on spring constant h_1. In (1), λu_t is called a weak damping term and we call $-\lambda\Delta u_t$ instead of λu_t a strong damping term. We also consider the following control function in (2) as a feedback control:

$$s(t) = -h_2 u_t(1,t) - h_3 \sin t, \quad t \geq 0, \tag{4}$$

where h_2 and h_3 are positive constants under the condition $\sin t < 2B(\| \nabla u(t) \|^2)\cos t$.

Mathematical models in mass production process, control engineering, and biological system are often governed by nonlinear Kirchhoff type equations. The purpose of this paper is to study the existence and uniqueness of solutions of the model system with mixed boundary conditions. Moreover stability problems which investigate decay estimates of energy for the model system, are given.

Recently, the important problem of vibration suppression of axially moving string-like continua has received attention by our results [10, 11, 12, 13]. Axially moving string is a typical model widely used to represent threads, wires, belt, magnetic tape, cables and band-saws, especially when the subject concerned is long and narrow enough. Several our results have derived and studied linear and nonlinear mathematical models which describe the movement of such systems [10, 11]. And also, some our result has derived and studied some engineering system with respect to boundary feedback control [12].

Its original equation is given by

$$\rho h \frac{\partial^2 u}{\partial t^2} = (p_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial u}{\partial x})^2 dx)\frac{\partial^2 u}{\partial x^2} \quad (5)$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time $t; E$, the young's modulus; ρ, the mass density; h, the cross section area; L, the length; and p_0, the initial axial tension. This equation was first introduced by Kirchhoff [14](See Carrier[5]); hence, (5) is known as the Kirchhoff-type equation. When $K = \lambda = \eta = 0$ and the Cauchy or mixed problem for (1) has been studied by many authors(see [18, 8]). In particular, many authors have investigated the nonlinear wave equation when $a(x) \equiv 1$ without the coriolis force term (i.e., $\eta = 0$) acting on the system (1)-(3)(see [18, 8, 21, 7, 2, 16, 17]).

On the other hand, in Chen et al.[6] investigated the equation

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial^2 u}{\partial t\, \partial x} + (\gamma^2 - 1)\frac{\partial^2 u}{\partial x^2} = \frac{3E}{2}\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^2, \tag{6}$$

where $u(x,t)$ is the transverse displacement at the axial coordinate x and time t; γ, the axial speed; and E, the Young's modulus (all dimensionless). Furthermore, Aassila and Kaya[1] investigated the system (1)-(3) with $a(x) \geq a_0 > 0$ and $a(x), a_x(x) \in L^\infty(\Omega)$, and the Dirichlet boundary condition without the coriolis force and forcing and dissipative terms. In case of system with mixed boundaries, some systems with various boundaries are studed by Bentsman and Hong[3], Vitillaro[20], Bociu and Lasiecka[4] and so on. In addition, Long[16] investigated the system (1)-(3) with $a(x) = 1$ and without the coriolis force. The authors mentioned above have only studied the local existence(no global existence) of solutions to their problems because of the nonlinearity of the term $a(x)B(\| \nabla u \|^2)$. And also, many researchers(See [3, 9]) investigated the asymptotic behavior of a solution for various practical systems by using some simulation.

The first objective of this paper is to verify the exponential decay for solutions to the system (1)-(3) with the boundary feedback control; For the guarantee of the energy decay of solutions to the system (1)-(3), we give the global existence result for the main system. Lastly, we try to show the system with controlled free boundary rather than without control has more stabilized vibration at the boundary by using simulation results.

II. Preliminaries

Throughout the paper, we will abbreviate to some notations $\Omega = (0,1)$, $T > 0$, $L^p = L^p(\Omega)$, $H^1 = H^1(\Omega)$, $H^2 = H^2(\Omega)$, where H^1, H^2 are the usual Sobolev spaces on Ω.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u:(0,T) \to X$ measurable, such that

$$\| u \|_{L^p(0,T;X)} = (\int_0^T \| u(t) \|_X^p \, dt)^{1/p} \quad \text{for} 1 \leq p < \infty,$$

and

$$\| u \|_{L^\infty(0,T;X)} = \underset{0<t<T}{esssup} \| u(t) \|_X \quad \text{for} p = \infty.$$

And also we put for some positive M,

$$W(M,T) = \{v \in L^\infty(0,T;H^2): v_t \in L^\infty(0,T;H^1(\Omega)), v_{tt} \in L^\infty(0,T;L^2(\Omega)), \quad (7)$$

$$\| v \|_{L^\infty(0,T;H^2(\Omega))} \leq M, \| v_t \|_{L^\infty(0,T;H^1)} \leq M, \| v_{tt} \|_{L^\infty(0,T;L^2)} \leq M, \quad (8)$$

$$|u(1,t)| \leq M, |u_t(1,t)| \leq M\}. \quad (9)$$

Lemma 2.1 $(b1, Theorem 6.2.1, p.137)$ *There exists the Hilbert orthonormal base* $\{\widetilde{w}_j\}$ *of* L^2 *consisting of the eigenfunctions* $\widetilde{w}_j$ *corresponding to the eigenvalue* λ_j *such that*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \lim_{j \to +\infty} \lambda_j = +\infty \quad (10)$$

Furthermore, the sequence $\{\widetilde{w}_j / \sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of H^1. On the other hand, we have also $\widetilde{w}_j$ satisfying the following boundary value problem:

$$-\Delta \widetilde{w}_j = \lambda_j \widetilde{w}_j, in \Omega, \quad (11)$$

$$a(1)\widetilde{w}_{jx} + h_1 \widetilde{w}_j(1) = 0, \widetilde{w}_j \in C^\infty(\overline{\Omega}). \quad (12)$$

Lemma 2.2 $(a1, Modified Imbedding theorem)$ *The embedding* $V^\circ C^0(\overline{\Omega})$ *is compact, where* $\| v \|_V = \| \nabla v(t) \|$ *and*

$$\| v \|_{C^0(\overline{\Omega})} \leq \sqrt{2} \| v \|_V, \quad for v \in V. (13)$$

The proof of Lemma 2.2 is also straightforward and we omit it.

III. The existence and uniqueness theorem of solution

We make the following assumptions:

$(A_1) h_1 \geq 0,\ K \geq 0,\ \lambda > 0,\ \eta \geq 0$ and $2\lambda \leq \eta$;

$(A_2) a(x) > 0$ for all $x \in \overline{\Omega}$, $a(x) \in L^\infty(\Omega), a_x(x) \in L^\infty(\Omega)$;

$(A_3) 0 < l_1 \leq essinf_{0 \leq x \leq 1} a(x), esssup_{0 \leq x \leq 1} a(x) \leq L_1, L_1(< +\infty) > 0$;

$(A_4) 0 < l_2 \leq essinf_{0 \leq x \leq 1} a_x(x), esssup_{0 \leq x \leq 1} a_x(x) \leq L_2, L_2(< +\infty) > 0$;

$(A_5) B \in C^1(\mathbb{R}_+), B(\tau) \geq b_0 > 0,\quad B'(\tau) < \delta$, where δ is a positive constant,

$0 < \tau \leq \| \nabla u(t) \|^2$ for $0 \leq t \leq T$;

$(A_6) K_0 = K_0(M, T) = \sup_{0 \leq \tau \leq M^2} B(\tau) > 0$ for $0 \leq t \leq T$.

Theorem 3.1 (Global Existence)
Let $B: [0, +\infty] \to [0, +\infty]$ satisfy the non-degeneracy condition $(i.e. B(\| \nabla u_0 \|^2) > 0)$. Let us assume that (A_1)-(A_6) hold and initial data $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$.

Then there exists a positive number M for every $T > 0$ such that the system (1)-(3) admits a unique global solution u in $W(M, T)$.

Proof. By using Galerkin's approximation, Lemma 2.1, Lemma 2.2 and a routine procedure similar to that of cite [10], we can the global existence result for the solution subject to (1)-(3) under the assumptions(A_1)-(A_6).

Remark 3.2 *In case of system using strong damping term instead of weak damping term, we can easily get the same result of solutions guaranteed by the boundedness of $R_m(t)$ which is using the above proof.*

Now we introduce an example to illustrate Theorem 3.1 as follows:

Example 3.3 *We consider the following nonlinear wave equation with spring boundary conditions*

$$u_{tt}(x,t) - \exp(x)(\| \nabla u \|^2)\Delta u(x,t) + u(x,t) + u_t(x,t) + u_{xt}(x,t) = 0$$
$$in \ \ (x,t) \in (0,1) \times (0, \infty),$$
$$u(0,t) = 0, \exp(1) u_x(1,t) = -u(1,t) + s(t) \ on \ (0, \infty),$$
$$u(x,0) = \exp\left(-64\left(x - \tfrac{1}{2}\right)^2\right), u_t(x,0) = 0 \ in \ (0,1),$$

where $\delta (> 0)$ is a constant. We can choose the suitable constants h_2, h_3 of $s(t)$ in (4).

Actually, the above example satisfies the assumptions (A_1)-(A_6) and the given conditions for existence. Therefore, its global unique existence is guaranteed by Theorem 3.1.

IV. Asymptotic Behavior

In this section, we study the asymptotic behavior of the generalized energy as $t \to +\infty$

$$F(t) = \frac{1}{2}\left[\| u_t(t) \|^2 + \int_0^1 a(x) B(\| \nabla u(t) \|^2) |\nabla u(x,t)|^2 dx + K \| u(t) \|^2\right] + \frac{h_1 b_0}{2} |u(1,t)|^2 + h_3 b_0 u(1,t) \sin t, \quad (14)$$

where u is the unique solution of the system (1)-(3) given by Theorem 3.1.

To continue the proof, we need to introduce three new functionals

$$E_0(t) = \frac{1}{2}\int_0^1 [|u_t(t)|^2 + |\nabla u(t)|^2] dx, \quad (15)$$

$$E(t) = \frac{1}{2}\int_0^1 [|u_t(t)|^2 + a(x) B(\| \nabla u(t) \|^2) |\nabla u(x,t)|^2 + K |u(t)|^2] dx, \quad (16)$$

$$F(t) = E(t) + \frac{h_1 b_0}{2} |u(1,t)|^2 + h_3 b_0 u(1,t) \sin t. \quad (17)$$

In general, we say that $E_0(t)$ is the kinetic energy and $E(t)$ is the energy including not only kinetic facts but also potential facts.

Theorem 4.1 (Energy Decay)

Let $q > \eta + L_1\delta \geq 0$ and $\lambda \geq \frac{L_2^2 K_0^2}{L_1\delta} \geq 0$, and suppose that every definition and hypothesis in the previous chapter holds. Then, the solution $u(x,t)$ of the system (1)-(3) satisfies the following energy decay estimates: There exists a positive constant C_4 such that

$$E(t) \leq \alpha_1 E_0(0)\exp\{-C_4 t\} for\ all\ t \geq 0.$$

Proof. Multiplying the first equation in the system (1)-(3) by u_t and applying the boundary condition (2), we have

$$\frac{1}{2}\frac{d}{dt} \| u_t(t) \|^2 + \int_0^1 a_x(x)B(\| \nabla u(t) \|^2)u_t(x,t)u_x(x,t)dx$$
$$+\frac{1}{2}\frac{d}{dt}\int_0^1 a(x)B(\| \nabla u(t) \|^2)u_x(x,t)u_x(x,t)dx$$
$$-\frac{1}{2}\int_0^1 a(x)(B(\| \nabla u(t) \|^2))' u_x(x,t)u_x(x,t)dx$$
$$+\frac{h_1}{2}B(\| \nabla u(t) \|^2)\frac{d}{dt}|u(1,t)|^2 + h_3 B(\| \nabla u(t) \|^2)\frac{d}{dt}(u(1,t)\mathrm{sin}t)$$
$$+h_2 B(\| \nabla u(t) \|^2)|u_t(1,t)|^2 - h_3 B(\| \nabla u(t) \|^2)u(1,t)\mathrm{cos}t$$
$$+\frac{K}{2}\frac{d}{dt} \| u(t) \|^2 + \lambda \| u_t(t) \|^2 + \eta\int_0^1 u_{xt}(x,t)u_t(x,t)dx = 0.$$

Dividing both sides by $B(\| \nabla u(t) \|^2)$ since $B(\| \nabla u(t) \|^2) > 0$, we get

$$\frac{1}{2B(\|\nabla u(t)\|^2)}\frac{d}{dt}\left[\| u_t(t) \|^2 + \int_0^1 a(x)B(\| \nabla u(t) \|^2)|\nabla u(x,t)|^2 dx + K \| u(t) \|^2\right]$$
$$+\frac{d}{dt}\left[\frac{h_1}{2}|u(1,t)|^2 + h_3 u(1,t)\mathrm{sin}t\right]$$
$$+\frac{1}{2B(\|\nabla u(t)\|^2)}\int_0^1 a_x(x)B(\| \nabla u(t) \|^2)u_t(x,t)u_x(x,t)dx$$
$$-\frac{1}{4B(\|\nabla u(t)\|^2)}\int_0^1 a(x)(B(\| \nabla u(t) \|^2))' u_x(x,t)u_x(x,t)dx$$
$$+h_2|u_t(1,t)|^2 - h_3 u(1,t)\mathrm{cos}t$$
$$+\frac{1}{2B(\|\nabla u(t)\|^2)}\left[\lambda \| u_t(t) \|^2 + \eta\int_0^1 u_{xt}(x,t)u_t(x,t)dx\right]$$
$$= 0.$$

From the assumptions (A_3)-(A_6) and the Cauchy-Schwarz inequality, we deduce that

$$\frac{1}{2B(\|\nabla u(t)\|^2)}\frac{d}{dt}\left[\| u_t(t) \|^2 + \int_0^1 a(x)B(\| \nabla u(t) \|^2)|\nabla u(x,t)|^2 dx + K \| u(t) \|^2\right]$$
$$+\frac{d}{dt}\left[\frac{h_1}{2}|u(1,t)|^2 + h_3 u(1,t)\mathrm{sin}t\right]$$
$$+\frac{1}{2B(\|\nabla u(t)\|^2)}\left[\left(\lambda - \varepsilon L_2 K_0 - \frac{\eta}{2}\right) \| u_t(t) \|^2 + \left(q - \frac{L_2 K_0}{2\varepsilon} - \frac{\eta}{2}\right) \| \nabla u(t) \|^2\right]$$
$$-\frac{1}{2B(\|\nabla u(t)\|^2)}\left(q + \frac{L_1\delta}{2} - \frac{L_2 K_0}{4\varepsilon}\right) \| \nabla u(t) \|^2$$
$$+\frac{1}{2B(\|\nabla u(t)\|^2)}h_3 u(1,t)\mathrm{sin}t \leq 0,$$

where $\mathrm{sin}t < 2B(\| \nabla u(t) \|^2)\mathrm{cos}t$.

Letting $\frac{L_2K_0}{2q-\eta} \le \varepsilon \le \frac{L_2K_0}{2(\eta+L_1\delta)}$ by the condition for q of Theorem 4.1 which is the positive constant and using the global existence results(i.e., $\| \nabla u(t) \| \le M$), we have

$$\frac{1}{2B(\|\nabla u(t)\|^2)}\frac{d}{dt}\left[\| u_t(t) \|^2 + \int_0^1 a(x)B(\| \nabla u(t) \|^2)|\nabla u(x,t)|^2dx + K \| u(t) \|^2\right]$$
$$+\frac{d}{dt}\left[\frac{h_1}{2}|u(1,t)|^2 + h_3u(1,t)\mathrm{sin}t\right] \quad (18)$$
$$+\frac{1}{2B(\|\nabla u(t)\|^2)}[C_1 \| u_t(t) \|^2 + C_2 \| \nabla u(t) \|^2 + h_3u(1,t)\mathrm{sin}t] \le 0,$$

where $C_1 = \lambda - \varepsilon L_2K_0 - \frac{\eta}{2}, C_2 = q - \frac{L_2K_0}{2\varepsilon} - \frac{\eta}{2}$ and C_1, C_2 are nonnegative constants from the assumptions of Theorem 4.1.

Multiplying (18) by the Kirchhoff part $B(\| \nabla u(t) \|^2)$, we get the following result from (A_5):

$$\frac{1}{2}\frac{d}{dt}\left[\| u_t(t) \|^2 + \int_0^1 a(x)B(\| \nabla u(t) \|^2)|\nabla u(x,t)|^2dx + K \| u(t) \|^2\right]$$
$$+\frac{1}{2}\frac{d}{dt}[h_1b_0|u(1,t)|^2 + h_3u(1,t)\mathrm{sin}t]$$
$$+\frac{C_3}{2}[\| u_t(t) \|^2 + \| \nabla u(t) \|^2 + h_3u(1,t)\mathrm{sin}t] \le 0,$$

where $C_3 = \min\{1, C_1, C_2\}$.

From (14)-(17), we deduce that

$$\frac{d}{dt}F(t) + \frac{C_3}{2}E_0(t) \le 0. \quad (19)$$

Proposition 1 (Energy equivalence)

$$\alpha_0E_0(t) \le F(t) \le \alpha_1E_0(t) for all t \ge 0,$$

where $\alpha_0 = \min\{1, l_1b_0 + K\}$ and $\alpha_1 = \max\{1, K, L_1K_0, b_0(h_1 + 1)M\}$.

Proof. By the assumptions (A_3) and (A_6), we have

$$\max\{1, K, L_1K_0, b_0(h_1 + 1)M\}E_0(t) \ge F(t).$$

And also, applying the assumptions (A_3) and (A_6), Lemma 2.2, and the positivity of $\frac{h_1b_0}{2}|u(1,t)|^2$, we deduce

$$\min\{1, l_1b_0 + K\}E_0(t) \le F(t).$$

From (19) and Proposition 1, we get

$$E_0(t) \le E_0(0)\exp\{-C_4t\} \text{for all } t \ge 0 \text{ and as } t \to +\infty,$$

where C_4 is a positive constant.

From Proposition 1, we get $F(t) \le \alpha_1E_0(t)$. This implies that

$$F(t) \le \alpha_1E_0(0)\exp\{-C_4t\} \quad \text{for all } t \ge 0 \text{ and as } t \to +\infty.$$

By the positivity of $\frac{h_1b_0}{2}|u(1,t)|^2$, we also deduce

$$E(t) \le \alpha_1E_0(0)\exp\{-C_4t\} \quad \text{for all } t \ge 0 \text{ and as } t \to +\infty.$$

V. Numerical Results

Now, we try to deal with numerical simulation results for the special system under some assumptions of Example 3.3 In the numerical results section, we consider two parts, that is, simulation of solution's shapes controlled system or not in time with respect to energy decay of solutions on the free boundary.

5.1 Solution's Shapes in time

When it comes to the numerical results of solutions, our purpose is showing solution for 3-dimension($i.e.$ $\mathrm{u(x,t),x,t}$) in the system. We consider that boundaries $\exp(1)u_x(1,\mathrm{t}) = -u(1,t) + s(\mathrm{t})$ at $x = 1$ in the system as we know. The system of special case makes with special boundary feedback control $-u_t(1,t) - \sin t$. Solution's shapes in full-time and $t = 1$ including boundaries are given in Figures 1

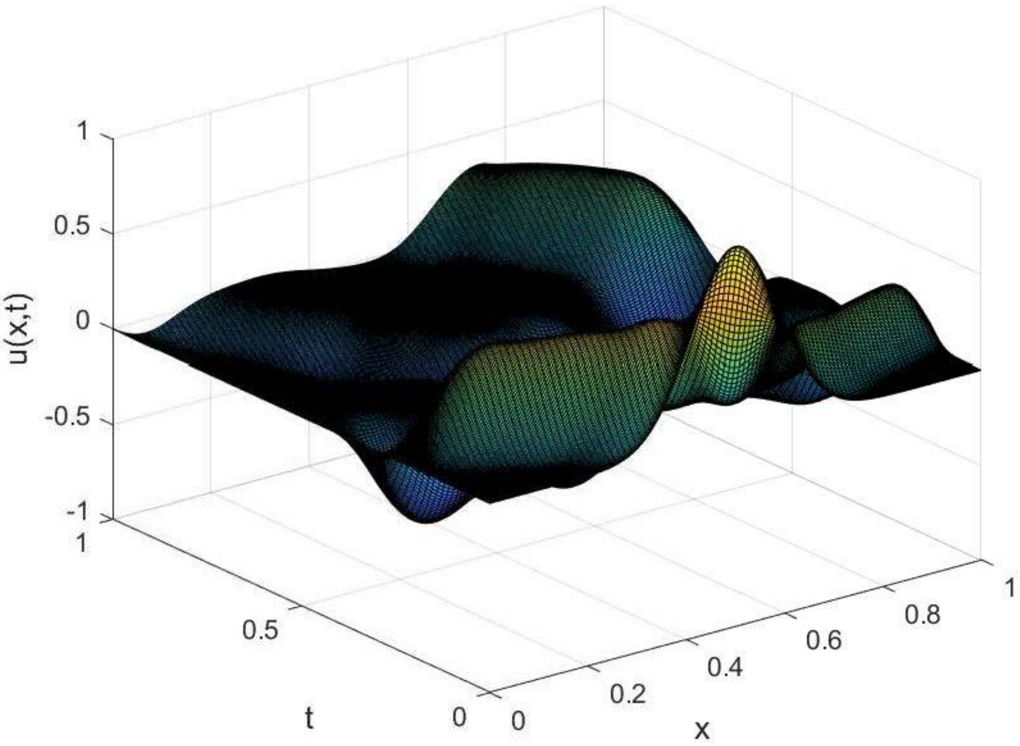

Figure 1: Solution's shapes on the whole time with spatial parts by boundary feedback control

5.2 Simulation of the main system with boundary controlled or not

In this section, we try to compare main system with boundary feedback control and main system with boundary feedback control. The aim of this section is showing the system with controlled free boundary rather than without control has more stabilized vibration at the boundary. The system with boundary feedback control (See Figure 1) has stable solution and relating energy unlike the system without boundary control (See Figure 2).

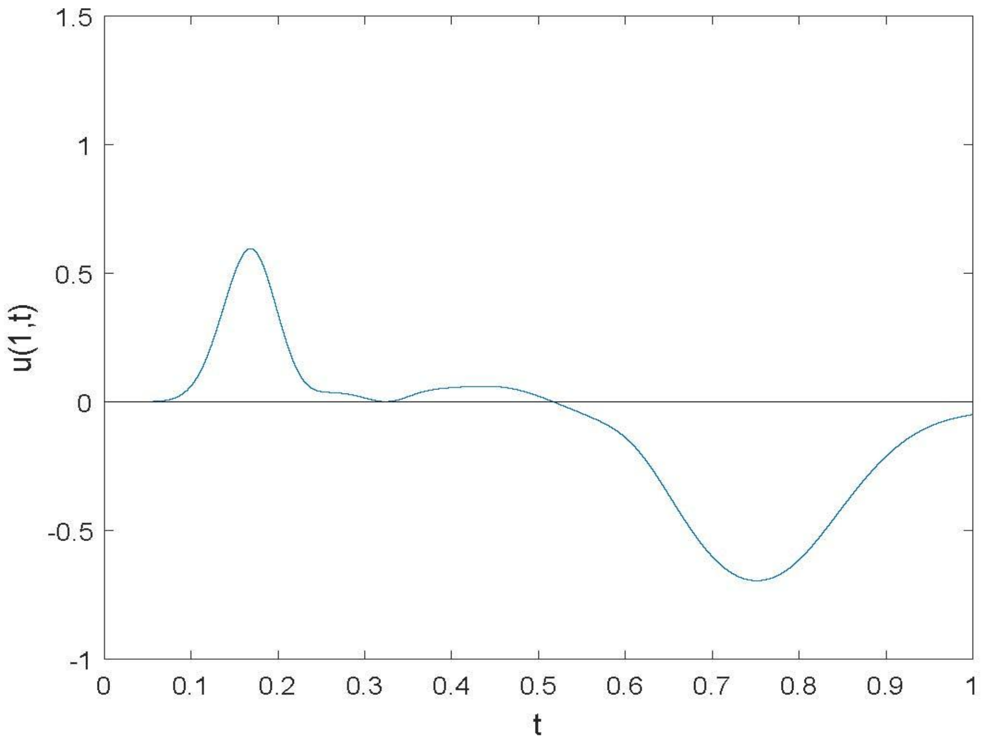

Figure 2: One sided spring boundary solution's shapes on the main system without boundary feedback control

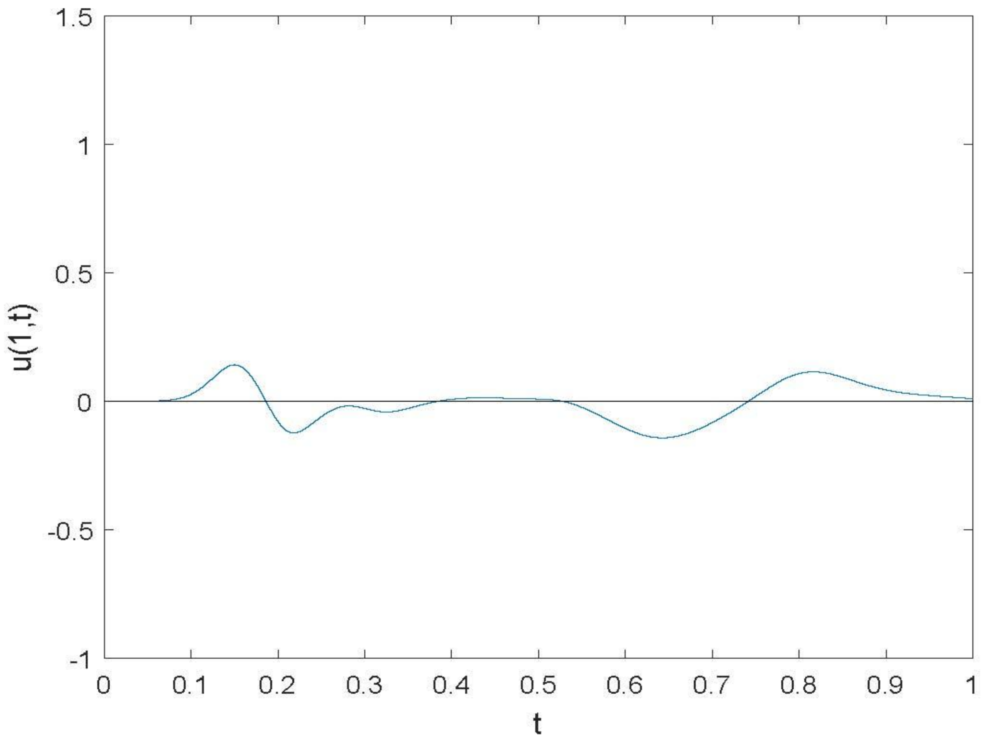

Figure 3: One sided spring boundary solution's shapes on the main system with boundary feedback control

In Section 5, for the numerical results, we used the standard finite difference method(FDM) and MATLAB.

VI. Conclusions

We dealt with analytical results and their numerical simulations. We established the global existence and uniqueness of weak solutions to this problem in time, and give an example and simulation to illustrate our results. Finally, we try to get the asymptotic behavior of energy and its simulation results. Actually, we get the result that the system with controlled free boundary rather than without control has more stabilized vibration at the boundary. These results are very useful, indeed, our results are able to apply industrial parts such as a typical model widely used to represent threads, wires, magnetic tapes, belts, band saws, and so on.

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