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A Comparative Study between Semi-Analytical Iterative Schemes for the Reliable Treatment of Systems of Coupled Nonlinear Partial Differential Equations.

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ABSTRACT

This study is aimed at making comparative analysis between three semi-analytical iterative methods viz: Adomian decomposition method, Temimi-Ansari method and Daftardar-Jafari method. The applicability and efficiency of these methods is confirmed by applying them to five different systems of nonlinear partial differential equations. Also, the respective errors of all the methods are ascertained to know the effectiveness and closeness to the exact solution. The results revealed, they all produce solution which is in good agreement with existing literature.

Keywords:Nonlinear PDEs, Iterative Schemes, Adomian decomposition method (ADM), Temimi- Ansari- method (TAM), Daftardar-Jafari Method (DJM)

I. INTRODUCTION

Systems of partial differential equations comprising both linear and nonlinear have attracted interest from academics the world over especially mathematicians and engineers for decades now and have been extensively studied for analytical and approximate solutions. These systems have numerous useful applications in chemical-reaction diffusion model of Brusselator, Physics, digital image processing, computer science, fluid mechanics, astrophysics, electronic, telecom, computer engineering, evolution equation applied in the propagation of shallow water waves [1-12].

Due to the important nature of these systems, several techniques or methods have been proposed to seek for closed form or numerical solutions. They includes, Homotopy Analysis Method (HAM), Homotopy perturbation method (HPM), Optimal Homotopy Asymptotic method (OHAM), Variational Iteration method (VIM), Differential Transform method (DTM), Laplace Adomian decomposition method (LADM), Aboodh Adomian decomposition method (AADM), Maghoud Adomian decomposition method (MADM), Modified Decomposition method (MDM), Riemann Invariants method (RIM), Fusion of Waveform Relaxation and Multigrid, Periodic multigrid wave form, Tanh-Coth method, Banach Contraction method (BCM), Lattice Boltzmann Method(LBM)[13-23].

Most of the methods produce closed form and approximate solutions but with much tedious computational work before giving solution, others experience inherent difficulty by way of calculating the so-called Adomian polynomial for the nonlinear term and some couldn't give accurate solution but a slowly converging series solution with marked error from the exact solution but time consuming. The Adomian decomposition method (ADM) was proposed by G. Adomian in 1994. This method requires expressing the unknown solution as an infinite decomposition series and the nonlinear term as an Adomian polynomial. This gives the solution elegantly in few steps of a rapidly converging series which matches the exact solution if it exists. The only snag in this method is in the calculation of the Adomian polynomials if the problem contains a nonlinear term. But this difficulty is easily overcomewith tact, experience, and diligence. It has been applied by many researchers to solve diverse problems. [24-33].

Equally, the novel Temimi-Ansari denoted (TAM) was presented by Temimi and Ansari to solve linear and nonlinear ODEs and PDEs. It's an iterative procedure which leads to an approximate solution that converges to the exact solution when the initial condition is satisfied. This method doesn't require linearization, perturbation, discretization, or calculation of Adomian polynomials which is a benchmark that reduce the complications in calculating the polynomial for nonlinear terms. For example, TAM has been applied to the following: Duffing Equation, KDV equation, Chemistry problems, nonlinear thin flow problems, Fokker-Plank's equation and many more. [34-40]. Most recently, the novelDaftardar-Jafari method (DJM) was developed by Daftardar-Gejji and Jafari in 2006 originally with an inspiration from the Banach contraction method. DJM has been extensively applied by academics to successfully solve linear and nonlinear PDEs, fractional dynamics, coupled Burger's equation, nonlinear Abel-type equation, Klein-Gordon equation [41--46].

In this paper, the motivation is to solve system of couplednonlinear partial differential equations, using the Adomian decomposition method (ADM), Temimi-Ansari method (TAM) and Daftardar-Jafari method (DJM). The methods are easy to implement and doesn't require lengthy calculations as is the case with other methods and less computational time. To confirm the efficiency and reliability of the iterative algorithms of the proposed methods, we solve four examples and compare their results. The work is organized as follows: In chapter 1, the relevant literatures and previous works using the proposed methods is presented in the introduction. Sections 2-4 is devoted to the fundamentals of the ADM, TAM and DJM. Numerical examples of nonlinear systems of PDEs are implemented with the different methods in section 5 and finally in section 6, the conclusion of the study is drawn.

II. FUNDAMENTALS OF ADOMIAN DECOMPOSITION METHOD (ADM)

Consider a general nonlinear differential equation of the form

$$F\big(u(x)\big) = g(x) \quad (1)$$

Where F is a nonlinear operator and y, g are both functions of x

Decomposing the nonlinear operator, F into the sum of linear and nonlinear operator.

$$L\big(u(x)\big) + R\big(u(x)\big) + N\big(u(x)\big) = g(x) \quad (2)$$

Where L is the highest order derivative that's invertible, R is the linear differential operator,N is a nonlinear term and g is the source term

Rewriting Eq. (2) for $L[u(x)]$ yield the form

$$L\big(u(x)\big) = g(x) - R\big(u(x)\big) - N\big(u(x)\big) \quad (3)$$

Taking the inverse operator, L^{-1} on both sides of Eq. (3), we get

$$y(x) = L^{-1}g(x) - L^{-1}R\big(u(x)\big) - L^{-1}N\big(u(x)\big)$$

$$y(x) = \phi - L^{-1}R\big(u(x)\big) - L^{-1}N\big(u(x)\big) \quad (4)$$

Where ϕ is the term arising from the integration of the source term using the boundary conditions

By the standard Adomian decomposition method, we write the unknown solution as an infinite decomposition series of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (5)$$

Putting Eq. (5) into Eq. (4), we obtain

$\sum_{n=0}^{\infty} u_n(x) = \phi - L^{-1}R(\sum_{n=0}^{\infty} u_n(x)) - L^{-1}N(\sum_{n=0}^{\infty} u_n(x))$ (6)

Matching both sides of Eq. (6), we obtain the zeroth order component given by

$$u_0 = \phi$$

Then the recursive relation is given by

$$u_{n+1}(x) = -L^{-1}R(u_n) - L^{-1}N(u_n), n \geq 0 \quad (7)$$

The solution of the problem in Eq. (1) is obtain as limit of the decomposing series

$$u(x) = \lim_{n\to\infty} u_n(x)$$
(8)

Similarly, the nonlinear term can be determined by an infinite series of the Adomian polynomials. That is,

$$N(u_0, u_1, u_2, \ldots, u_n) = \sum_{n=0}^{\infty} A_n \quad (9)$$

Then the $A_n^{'s}$ are obtained from the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty} \lambda^k y_k)]_{\lambda=0}, n = 0,1,2,3 \quad (10)$$

Using Eq. (9), the first five Adomian polynomials are given as

$$A_0 = N(u_0)$$

$$A_1 = u_1 N'(u_0)$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0)$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0)$$

$$A_4 = u_4 N'(u_0) + \frac{1}{2} N''(u_0)(2u_1u_3 + u_2^2) + \frac{1}{2} N'''(y_0) u_1^2 u_2 + \frac{1}{4!} N^{(iv)}(u_0) u_1^4$$

$$A_5 = u_5 N'(u_0) + \frac{1}{2} N''(u_0)(2u_1u_4 + 2u_2u_3) + \frac{1}{3!} N'''(u_0)(3u_1^2u_3 + 3u_1u_2^2) + \frac{4}{4!} N^{(iv)}(u_0)(u_1^3u_2) + \frac{1}{5!} N^{(v)}(u_0) u_1^5$$

$$A_6 = u_6 N'(u_0) + \frac{1}{2!} N''(u_0)(2u_1u_5 + 2u_1u_4 + u_3^2) + \frac{1}{3!} N'''(u_0)(3u_1^2u_4 + u_2^3 + 6u_1u_2u_3) + \frac{1}{4!} N^{(iv)}(u_0)(4u_1^3u_3 + 6u_1^2u_2^2) + \frac{5}{5!} N^{(v)}(u_0) u_1^4 u_2 + \frac{1}{6!} N^{(vi)}(u_0) u_1^6$$

$$A_7 = u_7 N'(u_0) + \frac{1}{2!} N''(u_0)(2u_1u_6 + 2u_2u_5 + 2u_3u_4) + \frac{1}{3!} N'''(u_0)(3u_1^2u_5 + 3u_1u_3^2 + 3u_3u_2^2 + 6u_1u_2u_4) + \frac{1}{4!} N^{(iv)}(u_0)(4u_1^3u_4 + 12u_1^2u_2u_3 + 4u_1u_2^3) + \frac{1}{5!} N^{(v)}(u_0)(5u_1^4u_3 + 10u_1^3u_2^2) + \frac{1}{6!} N^{(vi)}(u_0)u_1^5u_2 + \frac{1}{7!} N^{(vii)}(u_0)u_1^7$$

$$A_8 = u_8 N'(u_0) + \frac{1}{2!} N''(u_0)(2u_1u_7 + 2u_2u_6 + 2u_3u_5 + u_4^2) + \frac{1}{3!} N'''(u_0)(3u_1^2u_6 + 3u_4u_2^2 + 3u_2u_3^2 + 6u_1u_2u_5 + 6u_1u_3u_4) + \frac{1}{4!} N^{(iv)}(u_0)(4u_1^3u_5 + 12u_1^2u_2u_4 + 12u_1u_2^2u_3 + 6u_1^2u_3^2 + u_2^4) + \frac{1}{5!} N^{(v)}(u_0)(5u_1^4u_4 + 20u_1^3u_2u_3 + 10u_1^2u_2^3) + \frac{1}{6!} N^{(vi)}(u_0)(u_1^5u_3 + 15u_1^4u_2^2) + \frac{7}{7!} N^{(vii)}(u_0)u_1^6u_2 + \frac{1}{8!} N^{(viii)}(u_0)u_1^8$$

III. BASICS OF THE TEMIMI-ANSARI METHOD (TAM)

Consider the general differential equation in operator form as follows

$$L(y(x)) + N(y(x)) + f(x) = 0, \tag{11}$$

$$B\left(y, \frac{dy}{dx}\right) = 0, \text{ or } y_1(0) = \alpha \text{ and } y_1'(0) = \beta \tag{12}$$

Where x is the independent variable, $y(x)$ is an unknown function, $f(x)$ is a given known function, L is a linear operator, N is a nonlinear operator and B is a boundary operator.

To implement the TAM method, we first assume that$y_0(x)$ is an initial guess that satisfy the problem in Eq. (11) subject to Eq. (12).

$$L(y_0(x)) + f(x) = 0, B\left(y_0, \frac{dy_0}{dx}\right) = 0 \text{or } y_0(0) = \alpha \text{ and } y_0'(0) = \beta \tag{13}$$

The next approximate solution is obtained by solving the problem

$$L(y_1(x)) + N(y_0(x)) + f(x) = 0,\ B\left(y_1, \frac{dy_1}{dx}\right) = 0, \text{or } y_1(0) = \alpha \text{ and } y_1'(0) = \beta \tag{14}$$

The next iterate of the problem become

$$L(y_2(x)) + N(y_1(x)) + f(x) = 0,\ B\left(y_2, \frac{dy_1}{dx}\right) = 0, \text{or } y_2(0) = \alpha \text{ and } y_2'(0) = \beta \tag{15}$$

Continuing the same way to obtain the subsequent terms, the general equation of the method becomes

$$L(y_{n+1}(x)) + N(y_n(x)) + f(x) = 0, B\left(y_{n+1}, \frac{dy_{n+1}}{dx}\right) = 0, \text{ or } y_{n+1}(0) = \alpha \text{ and } y_{n+1}'(0) = \beta \tag{16}$$

Then the solution of the problem in Eq. (11) is given by

$y(x) = \lim_{n\to\infty} y_n(x)$ (17)

From Eq. (16), each $y(x)$ is considered alone as a solution for Eq. (11). This method easy to implement, straightforward and direct. The method gives better approximate solution which converges to the exact solution with only few members.

IV. DAFTARDAR-JAFARI METHOD (DJM)

Consider the function equation in Eq. (1) in the form

$L\big(y(x)\big) + N\big(y(x)\big) + g(x) = 0$ (18)

With the boundary or initial conditions

$B\left(y, \frac{dy}{dx}\right) = 0, y(0) = \alpha$ and $y'(0) = \beta$ (19)

Where x is the independent variable, $y(x)$ is the unknown function and $g(x)$ is given known function, N is a nonlinear operator, B is the boundary operator and $L(.) = \frac{d^n}{dx^n}$ is the linear operator.

Now applying the inverse operator, $L^{-1}(.) = \int_0^x \int_0^x (.)d\xi d\xi$ to Eq. (18) subject to Eq. (19), we obtain the equivalent form

$y(x) = \phi(x) + \int_0^x \int_0^x N\big(y(\xi)\big)d\xi d\xi$ (20)

Reducing the double integral in Eq. (20), we get the form

$y(x) = \phi(x) + \int_0^x (x-\xi)N\big(y(\xi)\big)d\xi d\xi$ (21)

Where $\phi(x)$ is an analytic function that comprises the sum of all initial conditions as well as the integration of the known function. $g(x)$

The solution of Eq. (21) takes the series form

$y(x) = \sum_{n=0}^{\infty} y_n(x)$ (22)

Now, using the nonlinear term we define the following terms

$G_0 = N(y_0)$

$G_m = N(\sum_{n=0}^{\infty} y_n) - N(\sum_{n=0}^{m-1} y_n), m \geq 1$

Next, we decompose $N(y)$ as follows

$N(\sum_{n=0}^{\infty} y_n) = \underbrace{N(y_0)}_{G_0} + \underbrace{[N(y_0+y_1) - N(y_0)]}_{G_1} + \underbrace{[N(y_0+y_1+y_2) - N(y_0+y_1)]}_{G_2} + \cdots$ (23)

In view of Eq. (23), we define a recurrence relation for the problem as follows.

$y_0 = f$ (24)

$y_1 = L(y_0) + G_0$ (25)

$y_{m+1} = L(y_m) + G_m, m \geq 1$ (26)

Using the linearity property, since L is a linear operator, we write

$\sum_{n=0}^{m} L(y_n) = L(\sum_{n=0}^{m} y_n)$, then we write

$$\sum_{n=1}^{m+1} y_n = \sum_{n=0}^{m} L(y_n) + N\left(\sum_{n=0}^{m} y_n\right)$$

$\sum_{n=1}^{m+1} y_n = L(\sum_{n=0}^{m} y_n) + N(\sum_{n=0}^{m} y_n), m \geq 1$ (27)

So that the recursive relation become

$\sum_{n=0}^{\infty} y_n = f + L(\sum_{n=0}^{\infty} y_n) + N(\sum_{n=0}^{\infty} y_n)$ (28)

Then the approximate solution of the problem follows

$y_n = \sum_{i=0}^{n} y_n$ (29)

V. NUMERICAL EXAMPLES

In this section, we apply the proposed semi-analytical iterative methods: Adomian decomposition method, Temimi-Ansari method and Daftardar-Jafari method to four systems of nonlinear partial differential equations to show the applicability, feasibility, and robustness of the methods.

Example 1 Consider the system of nonlinear partial differential equation

$u_t + vu_x + u = 1$

$v_t + uv_x - v = 1$ (30)

With initial conditions

$u(x,0) = e^x, v(x,0) = e^{-x}$ (31)

5.1 Implementation using ADM

Taking the inverse operator L_t^{-1} to both sides of the system in Eq. (30) using the initial conditions in Eq. (42) yield

$u(x,t) = e^x + t - L_t^{-1}(vu_x + u)$

$v(x,t) = e^{-x} + t + L_t^{-1}(uv_x + v)$ (32)

Now, we decompose the linear terms $u(x,t)$ and $v(x,t)$ as an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$ (33)

Similarly, the nonlinear terms, vu_x and uv_x are decomposed as Adomian polynomials

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n(x,t)$$

$N_2(x,t) = \sum_{n=0}^{\infty} B_n(x,t)$ (34)

Where A_n and B_n are called the Adomian polynomials. Plugging Eqs. (30) and (34) into Eq. (32), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = e^x + t - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right)$$

$\sum_{n=0}^{\infty} v_n(x,t) = e^{-x} + t + L_t^{-1}(\sum_{n=0}^{\infty} B_n(x,t) + \sum_{n=0}^{\infty} v_n(x,t))$ (35)

The recursive relations from the decomposition series become

$u_0(x,t) = e^x + t$

$u_{n+1}(x,t) = -L_t^{-1}(A_n + u_n(x,t)), n \geq 0$ (36)

and

$v_0(x,t) = e^{-x} + t$

$v_{n+1}(x,t) = L_t^{-1}(B_n + v_n(x,t)), n \geq 0$ (37)

5.2 Implementation by TAM

To solve the system in Eq. (32) using TAM, we proceed as follows

$L_1(u) = u_t, N_1(u,v) = vu_x + u$, and $f_1(x,t) = -1$

$L_2(v) = v_t, N_2(u,v) = -uv_x - u$, and $f_2(x,t) = -1$ (38)

The initial problem to be solved is of the form

$L_1(u_0(x,t)) + f_1(x,t) = 0, u_0(x,0) = e^x$ (39)

$L_2(v_0(x,t)) + f_2(x,t) = 0, u_0(x,0) = e^{-x}$ (40)

Applying the inverse operator to Eqs. (39) and (40), using the initial condition, gives

$u_0(x,t) = e^x + t$

$v_0(x,t) = e^{-x} + t$

The next approximate solution of the problem become

$L_1(u_1(x,t)) + N_1(u_0(x,t)) + f_1(x,t) = 0, u_1(x,0) = e^x$

$L_2(v_1(x,t)) + N_2(v_0(x,t)) + f_2(x,t) = 0, u_1(x,0) = e^{-x}$ (41)

Integrating both sides of Eq. (52) using the initial condition gives

$$\int_0^t u_{1t}(x,t)dt = \int_0^t \left[-\left(v_0(x,t)u_{0x}(x,t)\right) - u_0(x,t) + 1\right]dt$$
$$\int_0^t v_{1t}(x,t)dt = \int_0^t [u_0(x,t)v_{0x}(x,t) + v_0(x,t) + 1]dt \quad (42)$$

Solving the system in Eq. (42) gives the solutions of the form

$$u_1(x,t) = -2t + \frac{t^2}{2} + \left(1 + t + \frac{t^2}{2}\right)e^x$$
$$v_1(x,t) = 2t + \frac{t^2}{2} + \left(1 + t - \frac{t^2}{2}\right)e^{-x} \quad (43)$$

Next the third iterative solution of the problem become

$$L_1\left(u_2(x,t)\right) + N_1\left(u_1(x,t)\right) + f_1(x,t) = 0, u_2(x,0) = e^x$$
$$L_2\left(v_2(x,t)\right) + N_2\left(v_1(x,t)\right) + f_2(x,t) = 0, u_2(x,0) = e^{-x} \quad (44)$$

Applying the inverse operator to both sides of Eq. (44), we get

$$\int_0^t u_{2t}(x,t)dt = \int_0^t \left[-\left(v_1(x,t)u_{1x}(x,t)\right) - u_1(x,t) + 1\right]dt$$
$$\int_0^t v_{2t}(x,t)dt = \int_0^t [u_1(x,t)v_{1x}(x,t) + v_1(x,t) + 1]dt \quad (45)$$

Solving Eq. (56) yield the third iterative solution as

5.3 Implementation by DJM

Applying DJM to both sides of Eq. (32) and taking limits from 0 to t, we get the equation of the form.

$$u(x,t) = f_1(x,t) + \int_0^t (1 - vu_x - u)dt$$
$$v(x,t) = f_2(x,t) + \int_0^t (1 + uv_x + v)dt \quad (46)$$

Now, the nonlinear terms become

$$N(u_i) = \int_0^t (1 - v_i u_{ix} - u_i)dt$$
$$V(v_i) = \int_0^t (1 + u_i v_{ix} + v_i)dt \quad (47)$$

From the given initial conditions

$$f_1(x,t) = u_0(x,0) = e^x$$
$$f_2(x,t) = v_0(x,0) = e^{-x}$$

To obtain the preceding terms, we proceed as follows

$$u_1 = N(u_0) = \int_0^t (1 - v_0 u_{0x} - u_0)dt = -te^x$$
$$u_1 = -te^x$$
$$v_1 = N(v_0) = \int_0^t (1 + u_0 v_{0x} + v_0)dt \quad (48)$$
$$v_1 = \int_0^t (1 - 1 + e^{-x})dt = \int_0^t e^{-x}dt$$
$$v_1 = te^{-x}$$
$$\Rightarrow u_1 = -te^x, v_1 = te^{-x}$$

Now to get the next approximate solutions, we proceed as follows

$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$u_2 = \int_0^t te^x dt = \frac{t^2}{2!}e^x \quad (49)$$

Similarly, $v_2 = N(v_0 + v_1) - N(v_0)$

$$v_2 = \int_0^t te^{-x}dt = \frac{t^2}{2!}e^{-x}$$
$$\Rightarrow u_1 = \frac{t^2}{2!}e^x, v_1 = \frac{t^2}{2!}e^x$$

Continuing in the same way, we get the succeeding terms of the problems using the form

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1), v_2 = N(v_0 + v_1 + v_2) - N(v_0 + v_1)$$
$$\Rightarrow u_3 = \frac{t^2}{3!}e^x, v_3 = \frac{t^2}{3!}e^x \quad (50)$$

Thus, the solution of the problem gives

$$u(x,t) = \sum_{n=1}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

$$v(x,t) = \sum_{n=1}^{\infty} v_n = v_0 + v_1 + v_2 + \cdots \quad (51)$$

$$u(x,t) = e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^x . e^{-t}$$

$$v(x,t) = e^{-x} \left(1 + t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^{-x} . e^{t} \quad (52)$$

$$\Longrightarrow u(x,t) = e^{x-t}, v(x,t) = e^{-x+t}$$

Example 2. Consider the system of nonlinear partial differential equation

$$u_t + vu_x - 3u = 2$$

$$v_t - uv_x + 3v = 2 \quad (53)$$

Subject to the initial conditions

$$u(x,0) = e^{2x}, v(x,0) = e^{-2x} \quad (54)$$

5.4 Implementation using ADM

Taking the inverse differential operator, L_t^{-1} of bot sides of the system, we obtain

$$u(x,t) = e^{2x} - L_t^{-1}(vu_x + u)$$

$$v(x,t) = e^{-2x} + L_t^{-1}(uv_x - 3v) \quad (55)$$

Substituting the decomposition representation s for the linear and nonlinear terms, we get

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{2x} - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n\right)$$

$$\sum_{n=0}^{\infty} v_n(x,t) = e^{-2x} + L_t^{-1}\left(\sum_{n=0}^{\infty} B_n - 3\sum_{n=0}^{\infty} v_n\right) \quad (56)$$

Where A_n and B_n are the Adomian polynomials for the nonlinear terms, vu_x and uv_x respectively. Now the first three terms of the Adomian polynomials are given as follows

$$A_0 = v_0 u_{0x}$$

$$A_1 = v_1 u_{0x} + v_0 u_{1x} \quad (57)$$

$$A_2 = v_2 u_{0x} + v_1 u_{1x} + v_0 u_{2x}$$

Similarly, for uv_xwe obtain the following iterates

$$B_0 = u_0 v_{0x}$$

$$B_1 = u_1 v_{0x} + u_0 v_{1x} \quad (58)$$

$$B_2 = u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x}$$

Substituting the above Adomian polynomials into the recursive relation, we obtain the solution as follows.

$$(u_0, v_0) = (e^{2x}, e^{-2x})$$

$$(u_1, v_1) = (-2t - te^{2x}, -2t - 3te^{-2x}) \quad (59)$$

$$(u_2, v_2) = \left(\frac{5}{2}t^2 e^{2x} + 5t^2, 6t^2 + 5t^2 e^{-2x}\right)$$

Using the above iterative solution, the solution of problem in closed form become

$$(u, v) = (e^{2x+3t}, e^{-2x-3t}) \quad (60)$$

5.5 Implementation by TAM

Applying TAM to the above system, we proceed as follows

$$L_1(u) = u_t, N_1(u) = vu_x - 3u, f_1(x,t) = -2$$

$$L_2(v) = v_t, N_2(u) = -uv_x + 3v, f_2(x,t) = -2 \quad (61)$$

The initial problem to be solved is given by

$$L_1\big(u_0(x,t)\big) + f_1(x,t) = 0, u_0(x,0) = e^{2x}$$

$$L_2\big(v_0(x,t)\big) + f_2(x,t) = 0, v_0(x,0) = e^{-2x} \quad (62)$$

Applying the inverse operator, L_t^{-1} to both sides of the above equation, we get the form

$$\int_0^t u_{0t}(x,t)dt = \int_0^t 2dt$$

$$\int_0^t v_{0t}(x,t)dt = \int_0^t 2dt \quad (63)$$

In view of the above, the initial solution become
$(u_0, v_0) = (e^{2x} + 2t, e^{-2x} + 2t)$ (64)
The next approximate solution is obtain using the form
$L_1\big(u_1(x,t)\big) + N_1\big(u_0(x,t)\big) + f_1(x,t) = 0, u_1(x,0) = e^{2x}$
$L_2\big(v_1(x,t)\big) + N_2\big(v_0(x,t)\big) + f_2(x,t) = 0, v_2(x,0) = e^{-2x}$ (65)
Integrating the above and taking limits from 0 to t yield the equivalent form
$\int_0^t u_{1t}(x,t)dt = \int_0^t (-v_0 u_{0x} - 3u_0 + 2)dt$
$\int_0^t v_{1t}(x,t)dt = \int_0^t (u_0 v_{0x} - 3v_0 + 2)dt$ (66)
Solving the above equation gives the first iterative solution
$u_1(x,t) = -2t^2 e^{2x} - 3te^{2x} - 3t^2$
$v_1(x,t) = -2t^2 e^{2x} - 3te^{2x} - 3t^2$ (67)
Continuing in the same way, the subsequent terms are found and the solution in closed form become
$u(x,t) = e^{2x+t}$
$v(x,t) = e^{-2x-t}$ (68)

5.6 Implementation using DJM

Integrating both sides of the given system subject to the initial conditions, we obtain the form
$u(x,t) = f_1(x,t) + \int_0^t (2 - vu_x + 3u)dt$
$v(x,t) = f_2(x,t) + \int_0^t (2 + uv_x - 3v)dt$ (69)
Now, the nonlinear terms of the problem become
$N(u_k) = \int_0^t (2 - v_k u_{kx} + 3u_k)dt$
$N(v_k) = \int_0^t (2 + u_k v_{kx} - 3v_k)dt$ (70)
Using the given initial condition, we get
$f_1(x,t) = u_0(x,0) = e^{2x}$
$f_2(x,t) = v_0(x,0) = e^{-2x}$ (71)
To obtain the subsequent terms, we proceed as follows
$u_1 = N(u_0) = \int_0^t (2 - v_0 u_{0x} + 3u_0)dt$
$v_1 = N(v_0) = \int_0^t (2 + u_0 v_{0x} - 3v_0)dt$ (72)
Solving the above give the solution
$(u_1, v_1) = (3te^{2x}, -3te^{-2x})$
$u_2 = N(u_0 + u_1) - N(u_0)$ (73)
$u_2 = \frac{3}{2} t^2 e^{2x}$
Similarly, $v_2 = N(v_0 + v_1) - N(v_0)$ yield
$v_2 = -\frac{3}{2} t^2 e^{2x}$
In view of the above the closed form solution of the problem become
$u(x,t) = e^{2x+3t}$
$v(x,t) = e^{-2x-3t}$ (74)
Example 3. Solve the system of nonlinear partial differential equation for the unknowns
$u_t + u_x v_x - w_y = 1$
$v_t + v_x w_x + u_y = 1$
$w_t + w_x u_x - v_y = 1$ (75)
Subject to the initial conditions
$u(x,y,0) = x + y$
$v(x,y,0) = x - y$ (76)
$w(x,y,0) = -x + y$

5.7 Implementation using ADM

Taking the inverse operator, L_t^{-1} of both sides of the given system subject to the initial conditions yields

$$u(x,y,t) = x + y + t - L_t^{-1}(u_x v_x - w_y)$$
$$v(x,y,t) = x - y + t - L_t^{-1}(v_x w_x + u_y)$$
$$w(x,y,t) = -x + y + t - L_t^{-1}(w_x u_x + v_y) \quad (77)$$

Writing the linear terms as a decomposition series and nonlinear terms as Adomian polynomials, we obtain the recursive scheme as follows

$$\sum_{n=0}^{\infty} u_n(x,y,t) = x + y + t - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} w_n\right)$$
$$\sum_{n=0}^{\infty} v_n(x,y,t) = x - y + t - L_t^{-1}\left(\sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} u_n\right) \quad (78)$$
$$\sum_{n=0}^{\infty} w_n(x,y,t) = -x + y + t - L_t^{-1}\left(\sum_{n=0}^{\infty} C_n + \sum_{n=0}^{\infty} v_n\right)$$

Where A_n, B_n and C_n are called the so-called the Adomian polynomials

The first three terms of the Adomian polynomials are given below

$$A_0 = u_{0x} v_{0x}$$
$$A_1 = u_{1x} v_{0x} + u_{0x} v_{1x}$$
$$A_2 = u_{2x} v_{0x} + u_{1x} v_{1x} + u_{0x} v_{2x} \quad (79)$$
$$B_0 = v_{0x} w_{0x}$$
$$B_1 = v_{1x} w_{0x} + v_{0x} w_{1x}$$
$$B_2 = v_{2x} w_{0x} + v_{1x} w_{1x} + v_{0x} w_{2x} \quad (80)$$
$$C_0 = w_{0x} u_{0x}$$
$$C_1 = w_{1x} u_{0x} + w_{0x} u_{1x}$$
$$C_2 = w_{2x} u_{0x} + w_{1x} u_{1x} + w_{0x} u_{2x} \quad (81)$$

Putting the above nonlinear terms into the recursive relation, we obtain the iterative solution as follows

$$(u_0, v_0, w_0) = (x + y + t, x - y + t, -x + y + y)$$
$$(u_k, v_k, w_k) = (0,0,0), \ k \geq 1 \quad (82)$$

Consequently, the solution of the nonlinear system of nonlinear PDE is given by

$$(u_0, v_0, w_0) = (x + y + t, x - y + t, -x + y + y) \quad (83)$$

5.8 Implementation using TAM

Rewriting the system in the form, gives

$$L_1(u) = u_t, N_1(u) = u_x v_x - w_y, f_1(x,y,t) = -1$$
$$L_2(v) = v_t, N_2(v) = v_x w_x + u_y, f_2(x,y,t) = -1$$
$$L_3(v) = w_t, N_2(w) = w_x u_x - v_y, f_3(x,y,t) = -1 \quad (84)$$

The initial problem to be solved is given by

$$L_1(u_0(x,y,t)) + f_1(x,y,t) = 0, u_0(x,y,0) = x + y$$
$$L_2(v_0(x,y,t)) + f_2(x,y,t) = 0, v_0(x,y,0) = x - y$$
$$L_3(w_0(x,y,t)) + f_2(x,y,t) = 0, v_0(x,y,0) = -x + y \quad (85)$$

Applying the inverse operator and taking the limits of integration from 0 to t, we get the initial iterative solution.

$$u_0(x,y,t) = x + y + t$$
$$v_0(x,y,t) = x - y + t$$
$$w_0(x,y,t) = -x + y + t \quad (86)$$

The next iterate approximation to the problem is obtained using the form

$$L_1(u_1(x,y,t)) + N_1(u_0(x,y,t)) + f_1(x,y,t) = 0, u_1(x,y,0) = x + y$$
$$L_2(v_1(x,y,t)) + N_2(v_0(x,y,t)) + f_2(x,y,t) = 0, v_1(x,y,0) = x - y \quad (87)$$
$$L_3(w_1(x,y,t)) + N_3(w_1(x,y,t)) + f_2(x,y,t) = 0, w_1(x,y,0) = -x + y$$

Rearranging the above yield the equivalent form as follows

$$L_1(u_1(x,y,t)) = -N_1(u_0(x,y,t)) - f_1(x,y,t), u_1(x,y,0) = x + y$$

$L_2(v_1(x,y,t)) = -N_2(v_0(x,y,t)) - f_2(x,y,t), v_1(x,y,0) = x - y$ (88)
$L_3(w_1(x,y,t)) = N_3(w_1(x,y,t)) + f_2(x,y,t), w_1(x,y,0) = -x + y$

Applying the inverse operator to both sides of the above and taking limits from 0 to t, we get

$$\int_0^t u_{1t}(x,t)dt = \int_0^t (-u_{0x}v_{0x} + w_{0y} + 1)dt$$
$$\int_0^t v_{1t}(x,t)dt = \int_0^t (-v_{0x}w_{0x} - u_{0y} + 1)dt \quad (89)$$
$$\int_0^t v_{1t}(x,t)dt = \int_0^t (-w_{0x}u_{0x} - v_{0y} + 1)dt$$

$$u_1(x,y,t) = \int_0^t (-u_{0x}v_{0x} + w_{0y} + 1)dt$$
$$v_1(x,y,t) = \int_0^t (-v_{0x}w_{0x} - u_{0y} + 1)dt \quad (90)$$
$$w_1(x,y,t) = \int_0^t (-w_{0x}u_{0x} - v_{0y} + 1)dt$$

Plugging in the derivatives we obtain the first iterative solution

$$u_1(x,y,t) = t$$
$$v_1(x,y,t) = t \quad (91)$$
$$w_1(x,y,t) = t$$

Continuing in the same way, the solution converges to the exact solution in the form.

$$u(x,y,t) = x + y + t$$
$$v(x,y,t) = x - y + t \quad (92)$$
$$w(x,y,t) = -x + y + t$$

5.9 Implementation by DJM

Applying the DJM to the system subject to the initial condition, we obtain the form

$$u(x,y,t) = f_1(x,y,t) + \int_0^t (1 - u_x v_x + w_y)dt$$
$$v(x,y,t) = f_2(x,y,t) + \int_0^t (1 - v_x w_x - u_y)dt \quad (93)$$
$$w(x,y,t) = f_3(x,y,t) + \int_0^t (1 - w_x u_x + v_y)dt$$

Now, the nonlinear terms are given as follows using the routine as follows.

$$N(u_k) = \int_0^t (1 - u_{kx}v_{kx} + w_{ky})dt$$
$$N(v_k) = \int_0^t (1 - v_{kx}w_{kx} - u_{ky})dt \quad (94)$$
$$N(w_k) = \int_0^t (1 - w_{kx}u_{kx} + v_{ky})dt$$

Using the given initial condition, we obtain the solution for the second iterate

$$f_1(x,y,t) = x + y$$
$$f_2(x,y,t) = x - y \quad (95)$$
$$f_3(x,y,t) = -x + y$$

Plugging the values of $k = 0,1,2,3 \ldots$. We obtain the subsequent terms as follows.

$$u_1 = N(u_0) = \int_0^t (1 - u_{0x}v_{0x} + w_{0y})dt$$
$$v_1 = N(v_0) = \int_0^t (1 - v_{0x}w_{0x} - u_{0y})dt \quad (96)$$
$$w_1 = N(w_0) = \int_0^t (1 - w_{0x}u_{0x} + v_{0y})dt$$

Plugging in the derivatives above and integrating, we obtain the solution as

$$u_1(x,y,t) = t$$
$$v_1(x,y,t) = t \quad (97)$$
$$w_1(x,y,t) = t$$

The solution of the problem in closed form is given by

$$u(x,y,t) = x + y + t$$
$$v(x,y,t) = x - y + t \quad (98)$$
$$w(x,y,t) = -x + y + t$$

Example 4. Solve the system of nonlinear system of partial differential equations

$$u_t + v_x w_y - v_y w_x = -u$$
$$v_t + w_x u_y + w_y u_x = v \quad (99)$$
$$w_t + u_x v_y + u_y v_x = w$$

Subject to the initial conditions

$$u(x,y,0) = e^{x+y}$$
$$v(x,y,0) = e^{x-y} \quad (100)$$
$$w(x,y,0) = e^{-x+y}$$

5.10 Implementation by ADM

Taking the inverse operator, $L_t^{-1} = \int_0^t (.)dt$ of both sides of the above system, we get

$$u(x,y,t) = e^{x+y} - L_t^{-1}\left(v_x w_y - v_y w_x + u\right)$$
$$v(x,y,t) = e^{x-y} - L_t^{-1}\left(w_x u_y + w_y u_x - v\right) \quad (101)$$
$$w(x,y,t) = e^{-x+y} - L_t^{-1}\left(u_x v_y + u_y v_x - w\right)$$

Let the linear and nonlinear terms be decomposed as an infinite series and Adomian polynomials respectively.

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t)$$
$$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t) \quad (102)$$
$$w(x,y,t) = \sum_{n=0}^{\infty} w_n(x,y,t)$$

$$A_n = v_x w_y$$
$$B_n = v_y w_x$$
$$C_n = w_x u_y$$
$$D_n = w_y u_x \quad (103)$$
$$E_n = u_x v_y$$
$$F_n = u_y v_x$$

Plugging the above into the above system yield the recursive scheme in the form

$$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{x+y} - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} u_n\right)$$
$$\sum_{n=0}^{\infty} v_n(x,y,t) = e^{x-y} - L_t^{-1}\left(\sum_{n=0}^{\infty} C_n + \sum_{n=0}^{\infty} D_n - \sum_{n=0}^{\infty} v_n\right) \quad (104)$$
$$\sum_{n=0}^{\infty} w_n(x,y,t) = e^{-x+y} - L_t^{-1}\left(\sum_{n=0}^{\infty} E_n + \sum_{n=0}^{\infty} F_n - \sum_{n=0}^{\infty} w_n\right)$$

The first four terms of the Adomian polynomials are given below

$$A_0 = v_{0x} w_{0y}$$
$$A_1 = v_{1x} w_{0y} + v_{0x} w_{1y}$$
$$A_2 = v_{2x} w_{0y} + v_{1x} w_{1y} + v_{0x} w_{2y} \quad (105)$$
$$A_3 = v_{3x} w_{0y} + v_{2x} w_{1y} + v_{1x} w_{2y} + v_{0x} w_{3y}$$

$$B_0 = v_{0y} w_{0x}$$
$$B_1 = v_{1y} w_{0x} + v_{0y} w_{1x}$$
$$B_2 = v_{2y} w_{0x} + v_{1y} w_{1x} + v_{0y} w_{2x} \quad (106)$$
$$B_3 = v_{3y} w_{0x} + v_{2y} w_{1x} + v_{1y} w_{2x} + v_{0y} w_{3x}$$

$$C_0 = w_{0x} u_{0y}$$

$C_1 = w_{1x}u_{0y} + w_{0x}u_{1y}$
$C_2 = w_{2x}u_{0y} + w_{1x}u_{1y} + w_{0x}u_{2y}$
$C_3 = w_{3x}u_{0y} + w_{2x}u_{1y} + w_{1x}u_{2y} + w_{0x}u_{3y}$ (107)
$D_0 = w_{0y}u_{0x}$
$D_1 = w_{1y}u_{0x} + w_{0y}u_{1x}$
$D_2 = w_{2y}u_{0x} + w_{1y}u_{1x} + w_{0y}u_{2x}$ (108)
$D_3 = w_{3y}u_{0x} + w_{2y}u_{1x} + w_{1y}u_{2x} + w_{0y}u_{3x}$
$E_0 = u_{0x}v_{0y}$
$E_1 = u_{1x}v_{0y} + u_{0x}v_{1y}$
$E_2 = u_{2x}v_{0y} + u_{1x}v_{1y} + u_{0x}v_{2y}$ (109)
$E_3 = u_{3x}v_{0y} + u_{2x}v_{1y} + u_{1x}v_{2y} + u_{0x}v_{3y}$
$F_0 = u_{0y}v_{0x}$
$F_1 = u_{1y}v_{0x} + u_{0y}v_{1x}$
$F_2 = u_{2y}v_{0x} + u_{1y}v_{1x} + u_{0y}v_{2x}$ (110)
$F_3 = u_{3y}v_{0x} + u_{2y}v_{1x} + u_{1y}v_{2x} + u_{0y}v_{3x}$

Plugging the above polynomials into the recursive relations, we obtain the first iterative solution as

$(u_0, v_0, w_0) = (e^{x+y}, e^{x-y}, e^{-x+y})$ (111)

Similarly, for the second solution we put $k = 0,1,2..$ into the recursive schemes as follows

$u_1 = -L_t^{-1}(A_0 - B_0 + u_0)$
$v_1 = -L_t^{-1}(C_0 + D_0 - v_0)$ (112)
$w_1 = -L_t^{-1}(E_0 + F_0 - w_0)$
$u_2 = -L_t^{-1}(A_1 - B_1 + u_1)$
$v_2 = -L_t^{-1}(C_1 + D_1 - v_1)$ (113)
$w_2 = -L_t^{-1}(E_1 + F_1 - w_1)$

$u_3 = -L_t^{-1}(A_2 - B_2 + u_2)$
$v_3 = -L_t^{-1}(C_2 + D_2 - v_2)$ (114)
$w_3 = -L_t^{-1}(E_2 + F_2 - w_2)$

Evaluating the above schemes give subsequent solutions as follows

$(u_1, v_1, w_1) = (-te^{x+y}, te^{x-y}, te^{-x+y})$
$(u_2, v_2, w_2) = \left(-\frac{t^2}{2!}e^{x+y}, \frac{t^2}{2!}e^{x-y}, \frac{t^2}{2!}e^{-x+y}\right)$ (115)
$(u_3, v_3, w_3) = \left(-\frac{t^3}{3!}e^{x+y}, \frac{t^3}{3!}e^{x-y}, \frac{t^3}{3!}e^{-x+y}\right)$

Now, using the formula for the unknowns

$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t)$
$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t)$ (116)
$w(x,y,t) = \sum_{n=0}^{\infty} w_n(x,y,t)$

The exact or closed form solution of the coupled nonlinear system become

$u(x,y,t) = e^{x+y-t}$
$v(x,y,t) = e^{x-y+t}$
(117)
$w(x,y,t) = e^{-x+y+t}$

5.11 Implementation by TAM

Applying TAM to both sides of the system, we have

$L_1(u) = u_t, N_1(u) = v_x w_y - v_y w_x + u, f_1(x,y,t) = 0$

$L_2(v) = v_t, N_2(v) = w_x u_y + w_y u_x - v, f_2(x,y,t) = 0$ (118)
$L_3(w) = w_t, N_3(w) = u_x v_y + u_y v_x - w, f_3(x,y,t) = 0$
The initial problem to be solved is given by
$L_1\big(u_0(x,y,t)\big) + f_1(x,y,t) = 0$
$L_2\big(v_0(x,y,t)\big) + f_2(x,y,t) = 0$ (119)
$L_3\big(w_0(x,y,t)\big) + f_3(x,y,t) = 0$
Integrating both sides and taking limits from 0 to t yields the integrals of the form
$\int_0^t u_{0t}(x,y,t)dt = 0$
$\int_0^t v_{0t}(x,y,t)dt = 0$ (120)
$\int_0^t w_{0t}(x,y,t)dt = 0$
Solving the above system of integrals give the initial solution as
$u_0(x,y,t) = e^{x+y}$
$v_0(x,y,t) = e^{x-y}$ (121)
$w_0(x,y,t) = e^{-x+y}$
The second problem to be solved for the first iterate become
$L_1\big(u_1(x,y,t)\big) + N_1\big(u_0(x,y,t)\big) + f_1(x,y,t), u_1(x,y,0) = e^{x+y}$
$L_2\big(v_1(x,y,t)\big) + N_2\big(v_0(x,y,t)\big) + f_2(x,y,t), v_1(x,y,0) = e^{x-y}$ (122)
$L_3\big(w_1(x,y,t)\big) + N_3\big(w_1(x,y,t)\big) + f_3(x,y,t), w_1(x,y,0) = e^{-x+y}$
Taking the inverse operator of both sides subject to the initial condition yield the set of integral equations as follows.
$\int_0^t u_{1t}(x,y,t)dt = \int_0^t (-v_{0x}w_{0y} + v_{0y}w_{0x} - u_0)dt$
$\int_0^t v_{1t}(x,y,t)dt = \int_0^t (-w_{0x}u_{0y} - w_{0y}u_{0x} + v_0)dt$ (123)
$\int_0^t w_{1t}(x,y,t)dt = \int_0^t (-u_{0x}v_{0y} - u_{0y}v_{0x} + w_0)dt$
In view of the above, the second iterative solution of the problem follows
$u_1(x,y,t) = -te^{x+y}$
$v_1(x,y,t) = te^{x-y}$ (124)
$w_1(x,y,t) = te^{-x+y}$
The third iterative problem to be solved is given as follows
$L_1\big(u_2(x,y,t)\big) + N_1\big(u_1(x,y,t)\big) + f_1(x,y,t), u_2(x,y,0) = e^{x+y}$
$L_2\big(v_2(x,y,t)\big) + N_2\big(v_1(x,y,t)\big) + f_2(x,y,t), v_2(x,y,0) = e^{x-y}$ (125)
$L_3\big(w_2(x,y,t)\big) + N_3\big(w_2(x,y,t)\big) + f_3(x,y,t), w_2(x,y,0) = e^{-x+y}$
Taking the inverse operator of both sides and taking limits from 0 to t, we obtain
$u_2(x,y,t) = \int_0^t (-v_{1x}w_{1y} + v_{1y}w_{1x} - u_1)dt$
$v_2(x,y,t) = \int_0^t (-w_{1x}u_{1y} - w_{1y}u_{1x} + v_1)dt$ (126)
$w_2(x,y,t) = \int_0^t (-u_{1x}v_{1y} - u_{1y}v_{1x} + w_1)dt$
Plugging in the derivatives to be above integral equation, we obtain the iterative solution as follows
$u_2(x,y,t) = -\frac{t^2}{2!}e^{x+y}$
$v_2(x,y,t) = \frac{t^2}{2!}e^{x-y}$ (127)
$w_2(x,y,t) = \frac{t^2}{2!}e^{-x+y}$
Continuing in the same way, the converging series solutions of the unknowns become

$$u(x,y,t) = \left(1 - t + \frac{t^2}{2!} - \cdots\right)e^{x+y}$$
$$v(x,y,t) = \left(1 + t + \frac{t^2}{2!} + \cdots\right)e^{x-y} \quad (128)$$
$$w(x,y,t) = \left(1 + t + \frac{t^2}{2!} + \cdots\right)e^{-x+y}$$

The solution of the problem in closed form is given by

$$u(x,y,t) = e^{x+y-t}$$
$$v(x,y,t) = e^{x-y+t} \quad (129)$$
$$w(x,y,t) = e^{-x+y+t}$$

5.12 Implementation by DJM

Applying DJM to both sides of the coupled nonlinear system of the given PDEs, we get

$$u(x,y,t) = f_1(x,y,t) + \int_0^t (-v_x w_y + v_y w_x - u)dt$$
$$v(x,y,t) = f_2(x,y,t) + \int_0^t (-w_x u_y - w_y u_x + v)dt \quad (130)$$
$$w(x,y,t) = f_3(x,y,t) + \int_0^t (-u_x v_y - u_y v_x + w)dt$$

Where the nonlinear terms are given by

$$N(u_k) = \int_0^t (-v_{kx} w_{ky} + v_{ky} w_{kx} - u_k)dt$$
$$N(v_k) = \int_0^t (-w_{kx} u_{ky} - w_{ky} u_{kx} + v_k)dt \quad (131)$$
$$N(w_k) = \int_0^t (-u_{kx} v_{ky} - u_{ky} v_{kx} + w_k)dt$$

Using the given initial condition, we obtain

$$f_1(x,y,t) = u_0(x,y,0) = e^{x+y}$$
$$f_2(x,y,t) = v_0(x,y,0) = e^{x-y} \quad (132)$$
$$f_3(x,y,t) = w_0(x,y,0) = e^{-x+y}$$

Using the algorithm schemes for the subsequent terms, we obtain

$$u_1(x,y,t) = N(u_0) = \int_0^t (-v_{0x} w_{0y} + v_{0y} w_{0x} - u_0)dt$$
$$v_1(x,y,t) = N(v_0) = \int_0^t (-w_{0x} u_{0y} - w_{0y} u_{0x} + v_0)dt \quad (133)$$
$$w_1(x,y,t) = N(w_0) = \int_0^t (-u_{0x} v_{0y} - u_{0y} v_{0x} + w_0)dt$$

Substituting the derivatives into the above integrals, we obtain the second iterative solution in the form.

$$u_1(x,y,t) = -te^{x+y}$$
$$v_1(x,y,t) = te^{x-y} \quad (134)$$
$$w_1(x,y,t) = te^{-x+y}$$

The subsequent terms are obtained using the algorithms below

$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$v_2 = N(v_0 + v_1) - N(v_0) \quad (135)$$
$$w_2 = N(w_0 + w_1) - N(w_0)$$

In view of the above, the third iterative solution of the problem become

$$u_2(x,y,t) = -\frac{t^2}{2!}e^{x+y}$$
$$v_2(x,y,t) = \frac{t^2}{2!}e^{x-y} \quad (136)$$
$$w_1(x,y,t) = \frac{t^2}{2!}e^{-x+y}$$

Thus, the closed form of the nonlinear system of PDEs is given by

$$u(x,y,t) = e^{x+y-t}$$

$v(x,y,t)=e^{x-y+t}$ (137)
$w(x,y,t)=e^{-x+y+t}$

Example 5 Consider the coupled system partial differential equation as follows

$u_t=uu_x+vu_y$

$v_t=uv_x+vv_y$

Subject to the initial condition

$u(x,y,0)=x^2$

$v(x,y,0)=y$

5.13 **Implementation by ADM**

Taking the inverse operator of both sides of the system subject to the initial condition, we get

$$u(x,y,t)=x^2+L_t^{-1}(uu_x+vu_y)$$

$v(x,y,t)=y+L_t^{-1}(uv_x+vv_y)$ (138)

Writing the linear and nonlinear operators as infinite series an Adomian polynomials, we get

$$\sum_{n=0}^{\infty}u_n(x,y,t)=x^2+L_t^{-1}\left(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n\right)$$

$\sum_{n=0}^{\infty}v_n(x,y,t)=y+L_t^{-1}(\sum_{n=0}^{\infty}C_n+\sum_{n=0}^{\infty}D_n)$ (139)

Matching both sides of the equation above we get the zeroth order components of the schemes

$u_0(x,y,t)=x^2$

$v_0(x,y,t)=y$

The respective recursive schemes of the problem become

$u_{n+1}(x,y,t)=L_t^{-1}(A_n+B_n)$

$v_{n+1}(x,y,t)=L_t^{-1}(C_n+D_n)$ (140)

The first four terms of the nonlinear terms are given as follows

$A_0=u_0u_{0x}$

$A_1=u_1u_{0x}+u_0u_{1x}$

$A_2=u_2u_{0x}+u_1u_{1x}+u_0u_{2x}$ (141)

$A_3=u_3u_{0x}+u_2u_{1x}+u_1u_{2x}+u_0u_{3x}$

$B_0=v_0u_{0y}$

$B_1=v_1u_{0y}+v_0u_{1y}$

$B_2=v_2u_{0y}+v_1u_{1y}+v_0u_{2y}$ (142)

$B_3=v_3u_{0y}+v_2u_{1y}+v_1u_{2y}+v_0u_{3y}$

$C_0=u_0v_{0x}$

$C_1=u_1v_{0x}+u_0v_{1x}$

$C_2=u_2v_{0x}+u_1v_{1x}+u_0v_{2x}$ (143)

$C_3=u_3v_{0x}+u_2v_{1x}+u_1v_{2x}+u_0v_{3x}$

$D_0=v_0v_{0y}$

$D_1=v_1v_{0y}+v_0v_{1y}$

$D_2=v_2v_{0y}+v_1v_{1y}+v_0v_{2y}$ (144)

$D_3=v_3v_{0y}+v_2v_{1y}+v_1v_{2y}+v_0v_{3y}$

Plugging the above derivatives into the recursive relation and using the preceding terms, we obtain

$u_1(x,y,t)=L_t^{-1}(A_0+B_0)$

$v_1(x,y,t)=L_t^{-1}(C_0+D_0)$ (145)

Evaluating the above, we obtain the second iterative solution as follows.

$(u_1,v_1)=(2tx^3,ty)$

Similarly, the next iterative solution is obtain using the integrals

$u_2(x,y,t)=L_t^{-1}(A_1+B_1)$

$v_2(x,y,t)=L_t^{-1}(C_1+D_1)$ (146)

In view of the above, we get the third iterative solution as follows

$$(u_2, v_2) = \left(6t^2x^5, \frac{t^2}{2}x^2 + t^2y\right)$$

The final solution is obtained using the sum of partial series as follows

$$u(x,y,t) = x^2(1 + 2tx + 6t^2x^2 + \cdots)$$
$$v(x,y,t) = y(1 + t + t^2 + t^3 + \cdots) \qquad (147)$$

5.14 ImplementationbyTAM

To implement TAM, we first write the system in operator form as follows.

$$u_t - (uu_x + vu_y) = 0$$
$$v_t - (uv_x + vv_y) = 0 \qquad (148)$$
$$L(u) = u_t, N(u) = -(uu_x + vu_y), f_1(x,y,t) = 0$$
$$L(v) = v_t, N(v) = -(uv_x + vv_y), f_2(x,y,t) = 0$$

The first problem to be solved is given by

$$L(u_0(x,y,t)) = f_1(x,y,t), u_0(x,y,t) = x^2$$
$$L(v_0(x,y,t)) = f_2(x,y,t), v_0(x,y,t) = y \qquad (149)$$

Integrating both sides and taking limit from 0 to t subject to the initial conditions yields

$$u_0(x,y,t) = x^2$$
$$v_0(x,y,t) = y$$

The second iterative solution to be solved become

$$L(u_1(x,y,t)) + N(u_0(x,y,t)) = 0, u_1(x,y,t) = x^2$$
$$L(v_1(x,y,t)) + N(v_0(x,y,t)) = 0, v_1(x,y,t) = y \qquad (150)$$

Taking the inverse operator of both sides subject to the initial condition, we obtain

$$u_1(x,y,t) = \int_0^t (u_0u_{0x} + v_0u_{0y})dt$$
$$v_1(x,y,t) = \int_0^t (u_0v_{0x} + v_0v_{0y})dt \qquad (151)$$

Plugging the derivatives into the above and solving yields the solution as

$$(u_1, v_1) = (2tx^3, ty)$$

The third iterative solution is obtained by solved by solving the equation

$$L(u_2(x,y,t)) + N(u_1(x,y,t)) = 0, u_2(x,y,t) = x^2$$
$$L(v_2(x,y,t)) + N(v_1(x,y,t)) = 0, v_2(x,y,t) = y \qquad (152)$$

Integrating both sides of the above and taking limits from 0 to t yield the corresponding equations of the form.

$$u_2(x,y,t) = \int_0^t (u_1u_{1x} + v_1u_{1y})dt$$
$$v_2(x,y,t) = \int_0^t (u_1v_{1x} + v_1v_{1y})dt \qquad (153)$$

Solving the above integrals after putting the derivatives gives the solution

$$(u_2, v_2) = \left(6t^2x^5, \frac{t^3}{3}y\right)$$

Again, the fourth iterative solution of the problem become

$$L(u_3(x,y,t)) + N(u_2(x,y,t)) = 0, u_3(x,y,t) = x^2$$
$$L(v_3(x,y,t)) + N(v_2(x,y,t)) = 0, v_3(x,y,t) = y \qquad (154)$$

Taking the inverse operator of both sides subject to the initial condition give the integral equations

$$u_3(x,y,t) = \int_0^t (u_2u_{2x} + v_2u_{2y})dt$$
$$v_3(x,y,t) = \int_0^t (u_2v_{2x} + v_2v_{2y})dt \qquad (155)$$

Solving the above give the solution in the form

$$(u_3, v_3) = \left(36t^5x^9, \frac{t^7}{63}y\right)$$

Therefore, taking the partial sum of the iterative solutions, we get the closed form solution of the system as

$u(x,y,t) = x^2(1 + 2tx + 6t^2x^2 + \cdots)$

$v(x,y,t) = y(1 + t + t^2 + \cdots)$ (156)

5.15 **Implementation by DJM**

Applying DJM to both sides of the system subject to the initial conditions, we get the corresponding equations.

$$u(x,y,t) = f_1(x,y,t) + \int_0^t (uu_x + vu_y)dt$$

$$v(x,y,t) = f_2(x,y,t) + \int_0^t (uv_x + vv_y)dt \quad (157)$$

Where$f_1(x,y,t) = x^2,\ f_2(x,y,t) = y$

$$N(u_i) = \int_0^t (u_i u_{ix} + v_i u_{iy})dt$$

$$N(v_i) = \int_0^t (u_i v_{ix} + v_i v_{iy})dt \quad (158)$$

$u_0(x,y,t) = f_1(x,y,t) = x^2$

$v_0(x,y,t) = f_2(x,y,t) = y$

$$u_1 = N(u_0) = \int_0^t (u_0 u_{0x} + v_0 u_{0y})\, dt$$

$$v_1 = N(v_0) = \int_0^t (u_0 v_{0x} + v_0 v_{0y})\, dt \quad (159)$$

Solving the above integral equations, we get the second iterative solution as

$(u_1, v_1) = (2tx^3, ty)$

Now, the succeeding terms are obtained using the following relations

$u_2 = N(u_0 + u_1) - N(u_0)$

$v_2 = N(v_0 + v_1) - N(v_0)$ (160)

Evaluating the above yield the solution as

$$(u_2, v_2) = \left(t^2x^3, \frac{t^2}{2!}y\right)$$

Similarly, using the relation the next iterative solution is found

$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$

$v_2 = N(v_0 + v_1 + v_2) - N(v_0 + v_1)$ (161)

Solving the above, give the fourth iterative solution as

$$(u_3, v_3) = \left(\frac{t^3}{3}x^3, \frac{t^3}{3!}y\right)$$

Summing the above iterative solution for each step, the closed form solution of the problem become

$$u = x^2\left(1 + 2tx + 6t^2x^2 + \frac{t^3}{3}x^3\right)$$

$$v = y\left(1 + t + t^2 + \frac{t^3}{3}\right) \quad (162)$$

VI. CONCLUSION

In this research article, three semi-analytical iterative methods were proposed to solve nonlinear coupled system of partial differential equations subject to initial conditions. The applicability of these methods was demonstrated by successfully applying them tosolvefivesystems of NLPDEs. It was found that, the methods produce rapidly converging series solution which gives the exact solution. The rate of convergence of the results obtained when compared with existing literature are found to be in excellent agreement. TAM and DJM proved to be of significant improvement to the ADM, as they give the solution with less computational work and overcome the inherent hurdle of calculating Adomian polynomials for the nonlinear term. It is recommended these methods be used for used for other highly involved problems.

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